APPLIED QUANTUM MECHANICS GROUP

UNIVERSITÀ DEGLI STUDI DI MILANO
DIPARTIMENTO DI FISICA

ULTIMATE PRECISION:

DEVELOPMENTS IN QUANTUM ESTIMATION THEORY

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FIRST YEAR PHD STUDENT WORKSHOP

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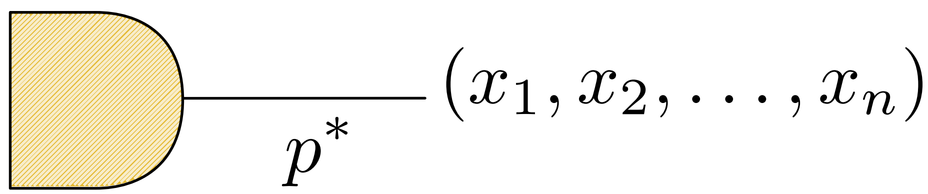
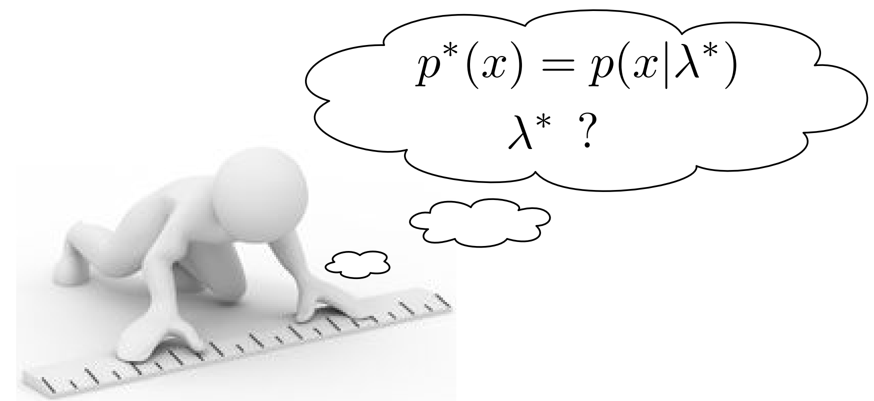
IN THIS TALK:

- Intro to Parameter Estimation Theory
    - classical estimation problem
    - quantum estimation problem
- Ultimate Precision Limits
    - Cramer - Rao theorems
- My Work
    - beyond the quantum Cramer - Rao

CLASSICAL PARAMETER ESTIMATION

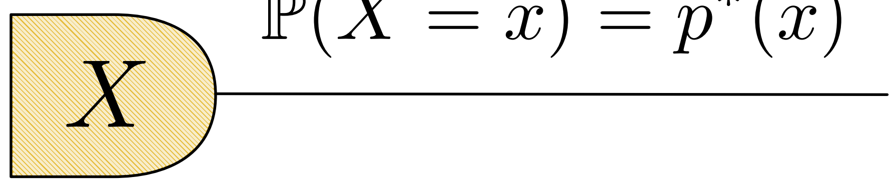




- an experimentalist has to gain knowledge on an information source by observations
- the information source is a *probability distribution*
- the experimentalist assumes the true probability distribution lives inside a *parametric model*

CLASSICAL ESTIMATION PROBLEM : INFER THE VALUE OF THE PARAMETER WHICH BETTER AGREES WITH THE OBSERVATIONS

$\mathbb{P}(X = x) = p^*(x)$



$(x_1, x_2, \ldots, x_n) \in \chi^n$

$p(x|\lambda)$ with $\lambda \in \Lambda$ and $p^*(x) = p(x|\lambda^*)$ — statistical model

X	random variable	$p^*(x)$	true probability density
χ	sample space	Λ	parameter space

estimator — unbiased — efficient

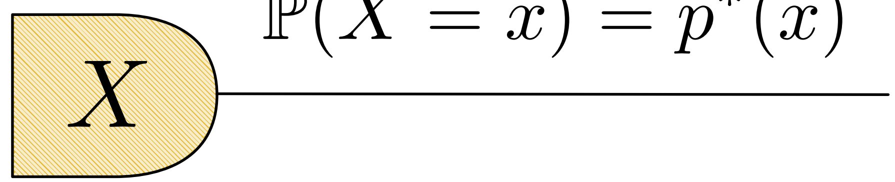
CLASSICAL ESTIMATION PROBLEM:

$\hat{\lambda} : \chi^n \to \Lambda$ s.t.

$\mathbb{E}(\hat{\lambda}) = \lambda^*$

$\mathrm{Var}(\hat{\lambda}) = \text{a minimum}$

$$\mathbb{P}(X = x) = p^*(x)$$



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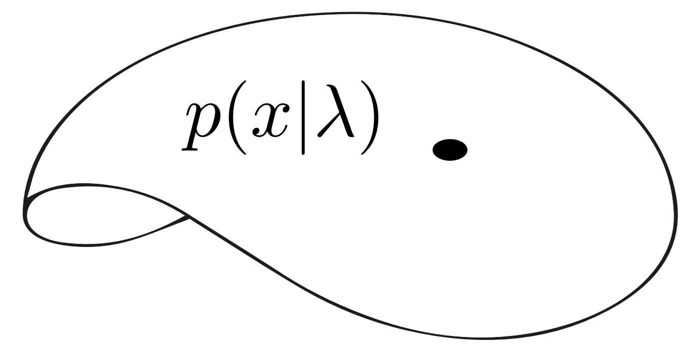
CLASSICAL ESTIMATION PROBLEM:

- does an unbiased, efficient estimator exist?
- what is its variance? ← best achievable precision

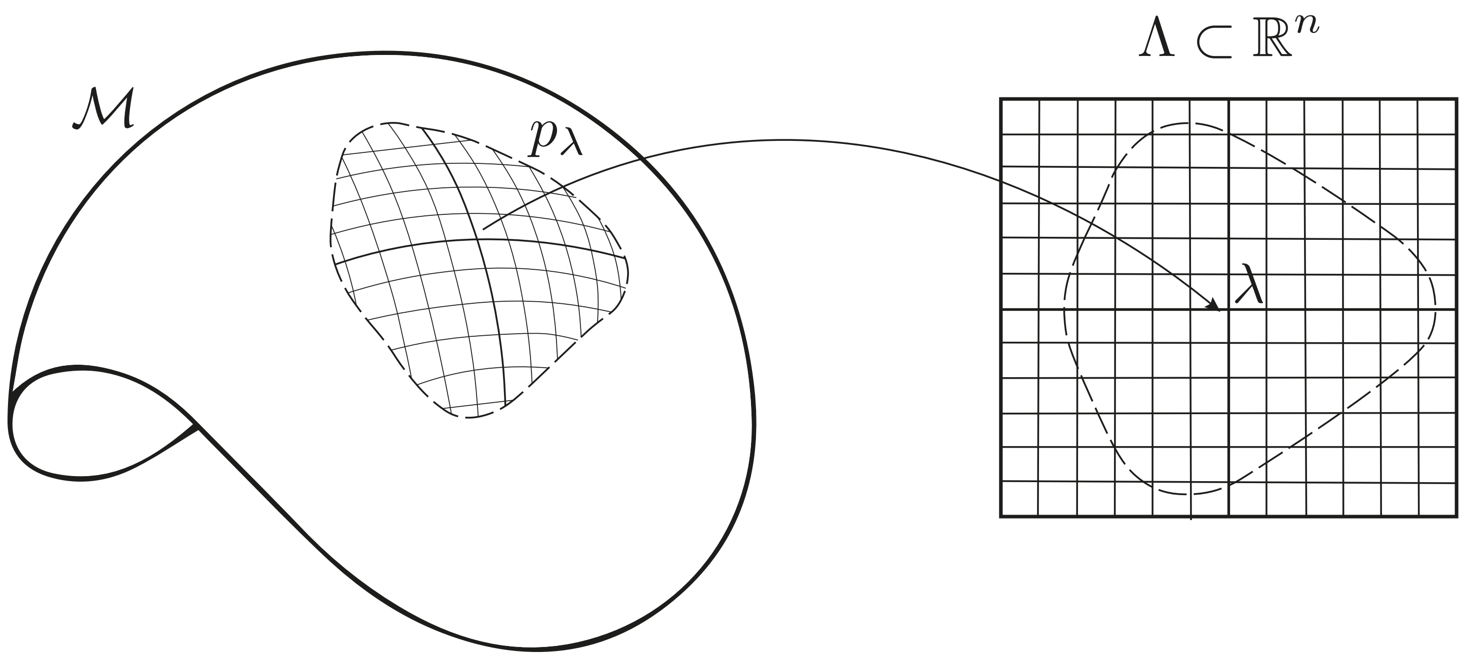
THE GEOMETRIC PICTURE

parametric model $\Longleftrightarrow$ statistical manifold

$$\{\, p(x|\lambda), \quad \lambda \in \Lambda \,\}$$



- each point is a probability density function
- Kullback - Leibler divergence measures $D_{KL}(q||p)$ difference between points
- Fisher Information metric $F_{ij}^{(X)}(\lambda)$ provides a natural way to measure lengths and angles



The parametrization $\lambda \to p(x|\lambda)$ provides *coordinate charts* on the manifold

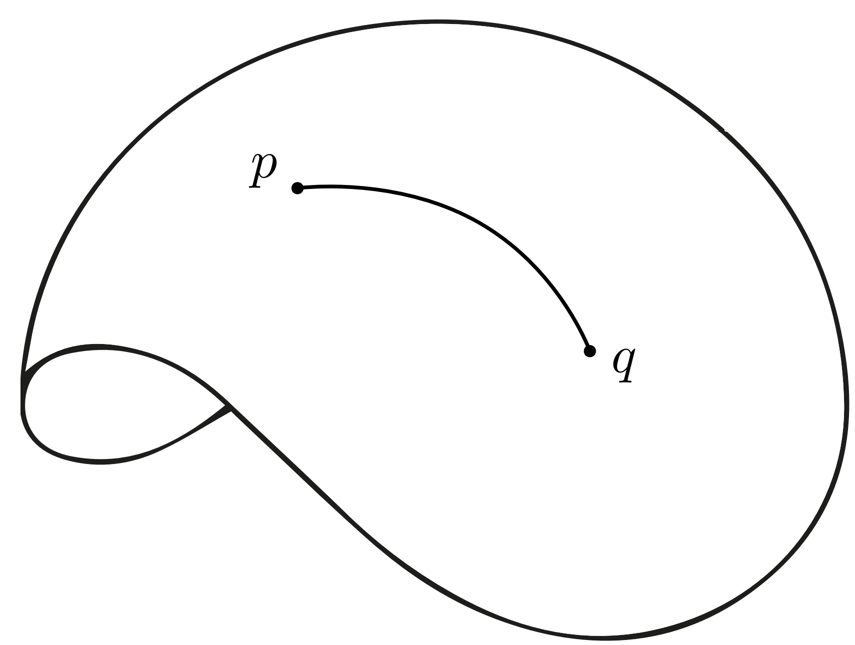
$$D_{KL}(p||q) = \int dx\, p(x) \log \frac{p(x)}{q(x)}$$

K-L divergence is a *premetric*:

- nonnegative
- $D_{KL}(p||q) = 0 \iff p = q$

but:

- asymmetric
- violates triangle-inequality



$$\partial_\lambda \, D_{KL}(\, p(x|\lambda) \,||\, p(x|\lambda^*) \,) \,|_{\lambda=\lambda^*} = 0$$

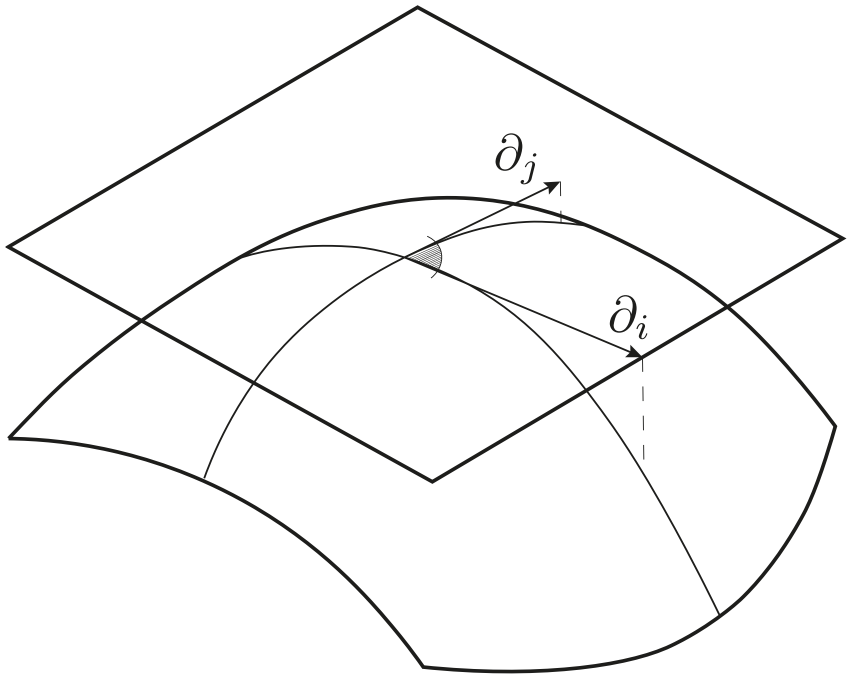
$$\Downarrow$$

$$D_{KL}(\, p(x|\lambda) \,||\, p(x|\lambda^*) \,) = 0 + 0 \cdot \lambda + \frac{1}{2} F_{ij}(\lambda^*) \lambda_i \, \lambda_j$$

FISHER INFORMATION METRIC

$$F^{(X)}_{ij}(\lambda^*) = \partial_i \partial_j \, D_{KL}(\, p(x|\lambda) \,||\, p(x|\lambda^*) \,) \,|_{\lambda=\lambda^*}$$

$$= \int dx \; p(x|\lambda^*) \, \partial_i \log p(x|\lambda^*) \, \partial_j \log p(x|\lambda^*)$$



$F^{(X)}_{ij}(\lambda)$ defines a *Riemannian metric* on $\mathcal{M}$

QUANTUM PARAMETER ESTIMATION

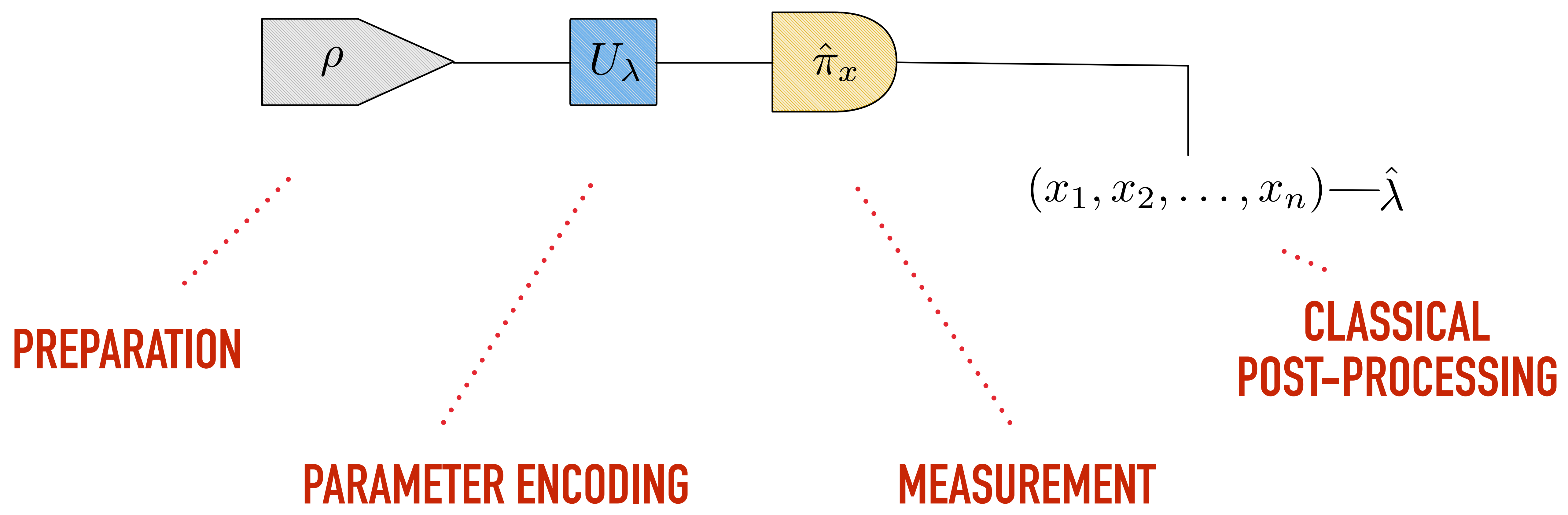
- a quantum system is represented by a *density operator* ρ
- an observable X is represented by a *Hermitian operator* $\hat{X} = \sum x\,\hat{\pi}_x$ so that $\mathbb{P}(X = x) = \mathrm{tr}(\rho\,\hat{\pi}_x)$

BUT HOW TO LEARN THE VALUE OF A NON-OBSERVABLE λ?

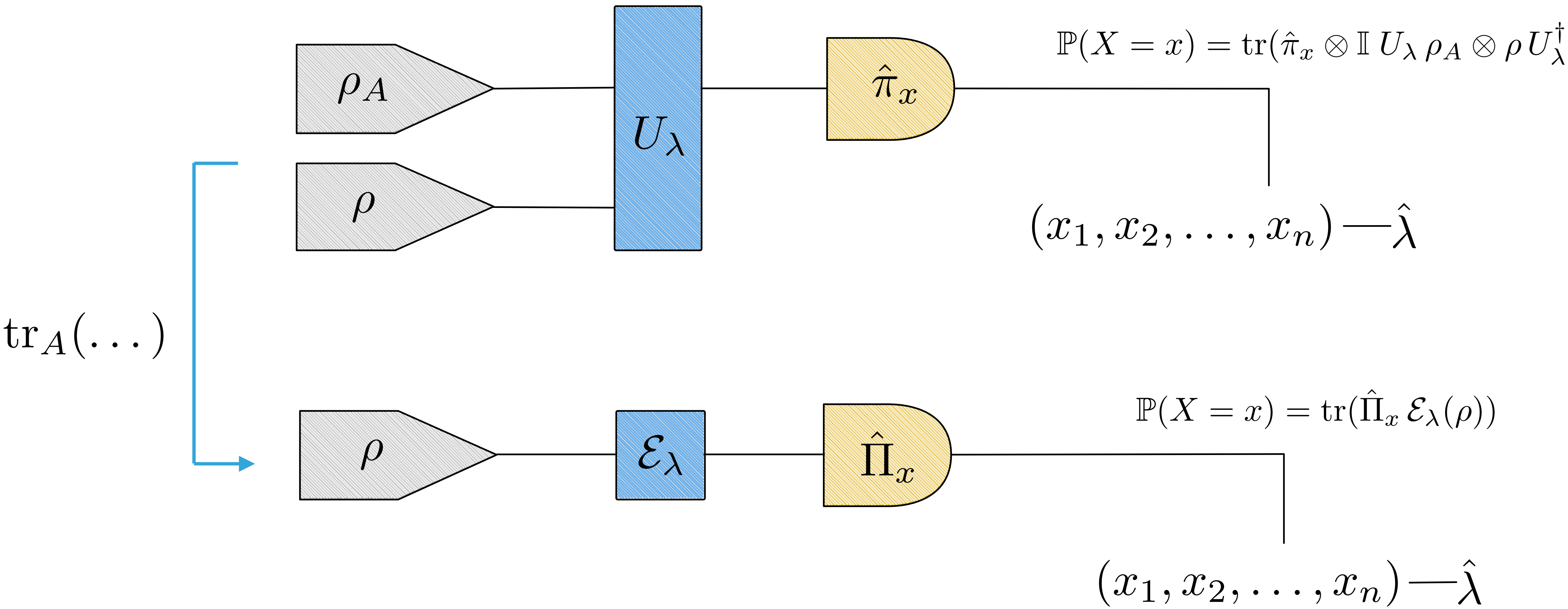
- many quantities of interest for quantum technologies do not correspond to an observable in the traditional sense:
  - a unitary phase (e.g. Mach-Zender interferometry) $\quad \rho \to e^{-i\lambda\,\hat{G}}\,\rho\,e^{i\lambda\,\hat{G}}$
  - a Hamiltonian parameter (e.g. a coupling constant) $\quad \rho \to e^{-i\hat{H}(\lambda)\,t}\,\rho\,e^{i\hat{H}(\lambda)\,t}$
  - a thermodynamical state parameter (e.g. temperature) $\quad \rho = \dfrac{e^{-\lambda\hat{H}}}{Z}$
- in all these cases one ends up with a *parametric* family of density operators

in QM one is limited to measure *observables*. But, chosen a specific observable $\hat{X}$ to measure, the probability distribution of the outcomes depends parametrically on λ. One thus is reduced to solving a *classical estimation problem*.

DIRECT MEASUREMENT



INDIRECT MEASUREMENT



quantum map → measurement operators →

$$\mathcal{E}_\lambda(\rho) = \sum_x \hat{M}_x\, \rho\, \hat{M}_x^\dagger$$

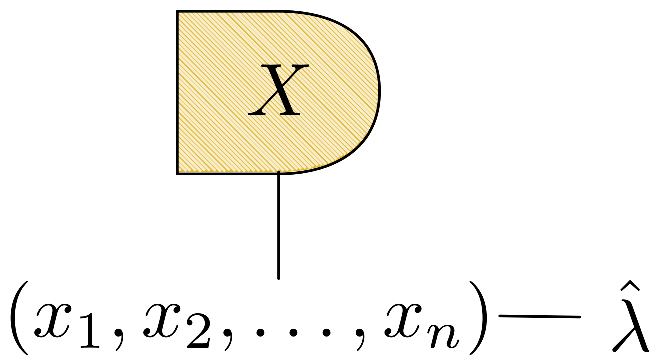
POVM → $\hat{\Pi}_x = \hat{M}_x^\dagger \hat{M}_x$

$$\sum_x \hat{\Pi}_x = \mathbb{I}\,, \qquad \hat{\Pi}_x > 0$$

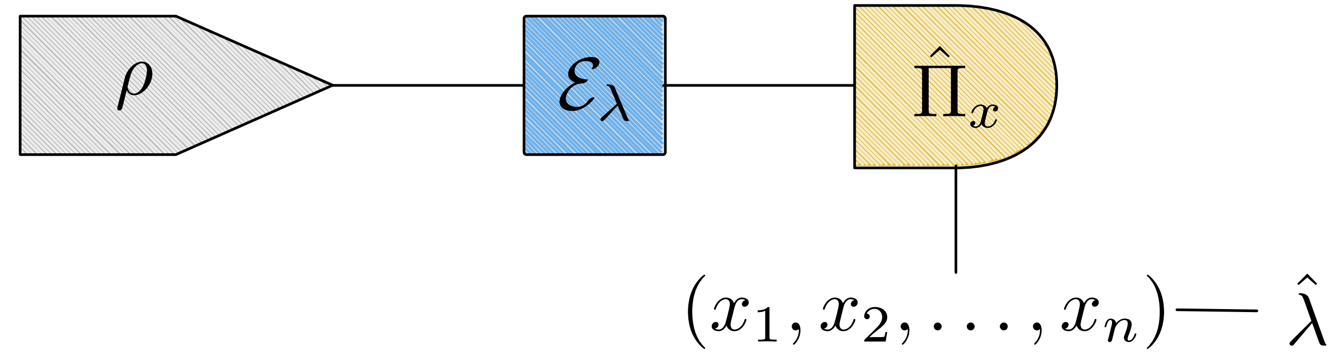
$\hat{\Pi}_x^\dagger = \hat{\Pi}_x$ but $\hat{\Pi}_x \hat{\Pi}_y \neq \hat{\Pi}_x\, \delta_{xy}$

CRAMER – RAO THEOREMS

classical



quantum



Achieving the *ultimate precision* requires minimization of $\mathrm{Var}(\hat{\lambda})$:

- choice of the quantum measurement scheme
- choice of the classical estimator

hard, fundamental problem of quantum metrology

easy, old problem in classical statistics

- choice of the classical estimator

CLASSICAL CRAMER - RAO THEOREM

GEOMETRY  STATISTICS

$$\mathrm{Var}(\hat{\lambda}) \geq \frac{1}{F(\lambda)} \, ,$$

moreover the bound can always be saturated, at least asymptotically

- choice of the quantum measurement scheme
  - maximize the Fisher Information over the set of all possible POVMs
  - the Fisher Information of the optimal measurement scheme is the ultimate precision allowed by QM

$$\max_{\{\Pi_x\}} F^{(X)}(\lambda) = ?$$

QUANTUM CRAMER - RAO THEOREM

S. L. Braunstein, C. M. Caves, Phys. Rev. Lett. 72, 3439 (1994)

Quantum Fisher Information

$$F^{(X)}(\lambda) \leq J(\lambda),$$

moreover the bound can always be saturated by a projective measurement of L_λ

symmetric logarithmic derivative

$$J(\lambda) = \mathrm{tr}(\mathcal{E}_\lambda(\rho) L_\lambda^2)$$

$$\partial_\lambda \mathcal{E}_\lambda(\rho) = \frac{1}{2}\left\{\mathcal{E}_\lambda(\rho), L_\lambda\right\}$$

$J(\lambda)$:

generalized uncertainty relations

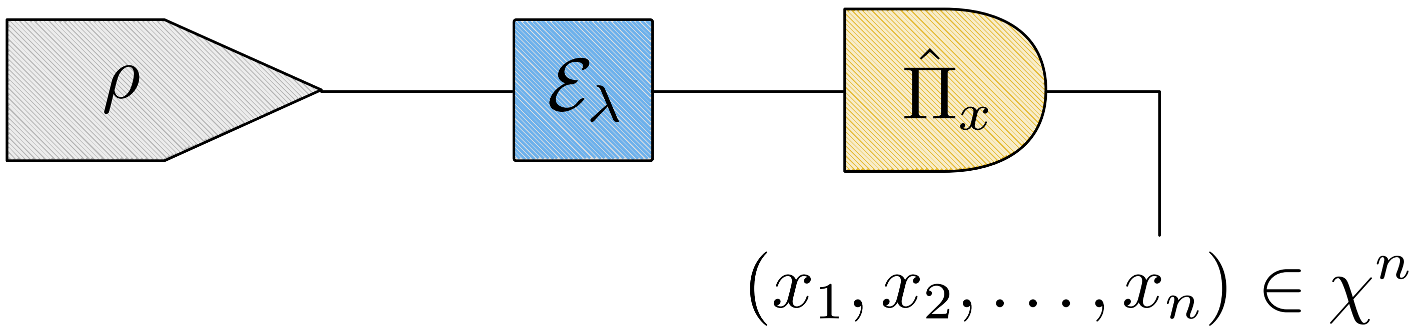
- quantifies the ultimate limits to precision allowed by QM
- defines a Riemannian metric on the manifold of quantum states
- is fundamental for the design and characterization of systems operating at the quantum scale

quantum metrology,
quantum sensing…

L. Seveso, M. A. C. Rossi, M. G. A. Paris, arXiv:1605.08653 (2016)

BEYOND THE QUANTUM CRAMER – RAO

the unknown parameter influences neither the sample space of outcomes nor the measurement aimed at extracting information on the parameter itself



- this assumption is *crucial* to introduce the quantum Fisher information as an upper bound to the Fisher information of any possible measurement
- however, there are relevant estimation problems where the assumption does not hold
- genuine ultimate bound to precision of quantum measurements?

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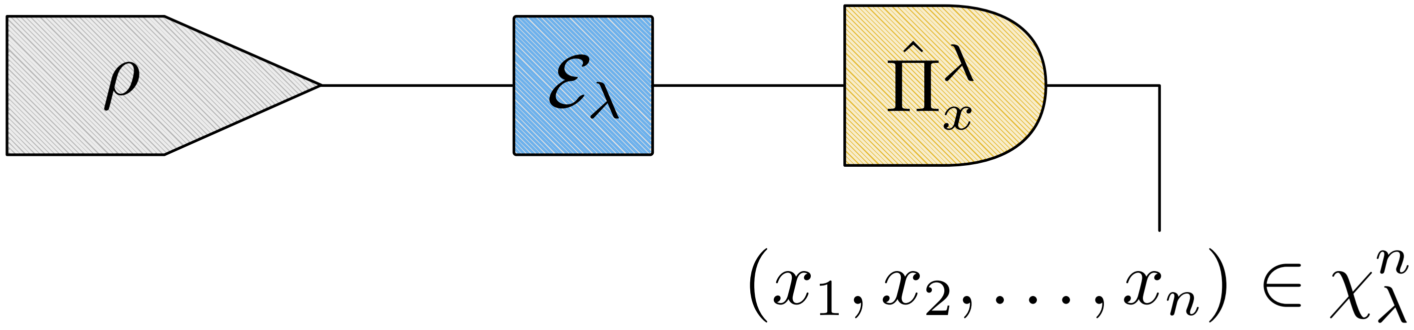

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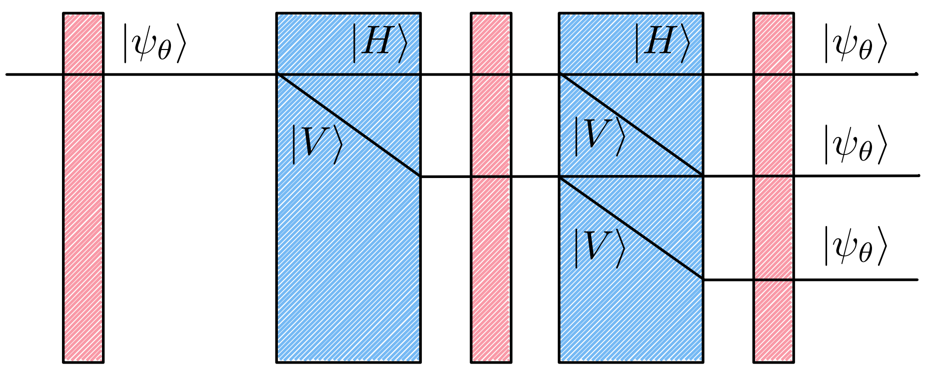
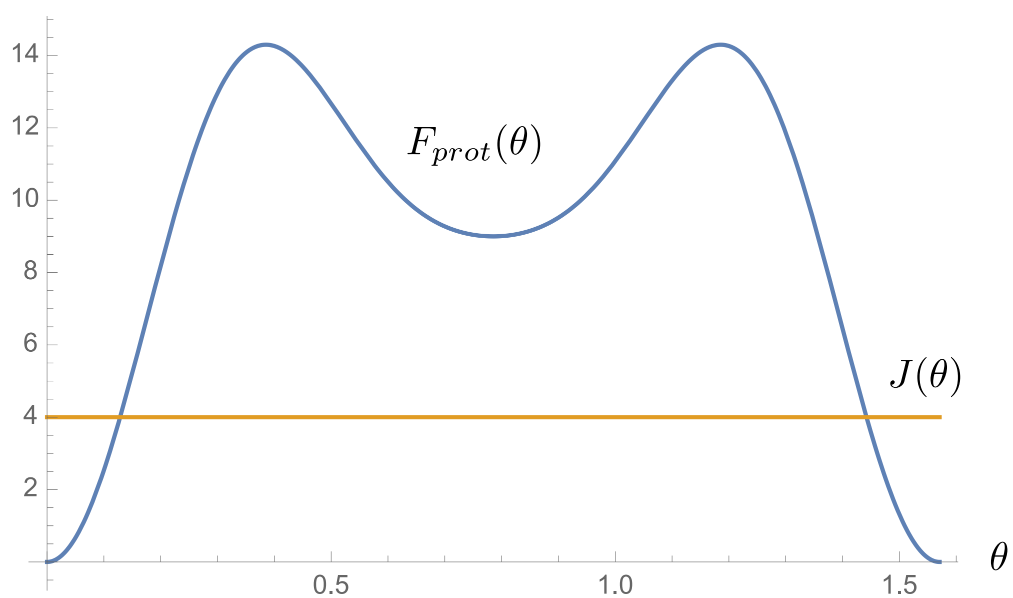
MY WORK

THE GOOD

measurement strategies that depend on the unknown parameter can lead to *better precision*



polarization estimation with parameter-dependent protective measurements



THE BAD

the link between geometry and statistics is severed

metric properties of statistical manifold

precision limits on the unknown parameter

TAKE - HOME MESSAGES

- quantum estimation theory based on the B - C inequality is the basis of quantum technology and has appealing geometric interpretation
- … but it only takes into consideration the information on the parameter coming from the manifold of states
- in fact, B - C inequality is violated when implementing measurement strategies that intrinsically depend on the parameter
  - new conceptual difference between classical and quantum parameter estimation
  - quantum measurement can be more precise than previously thought

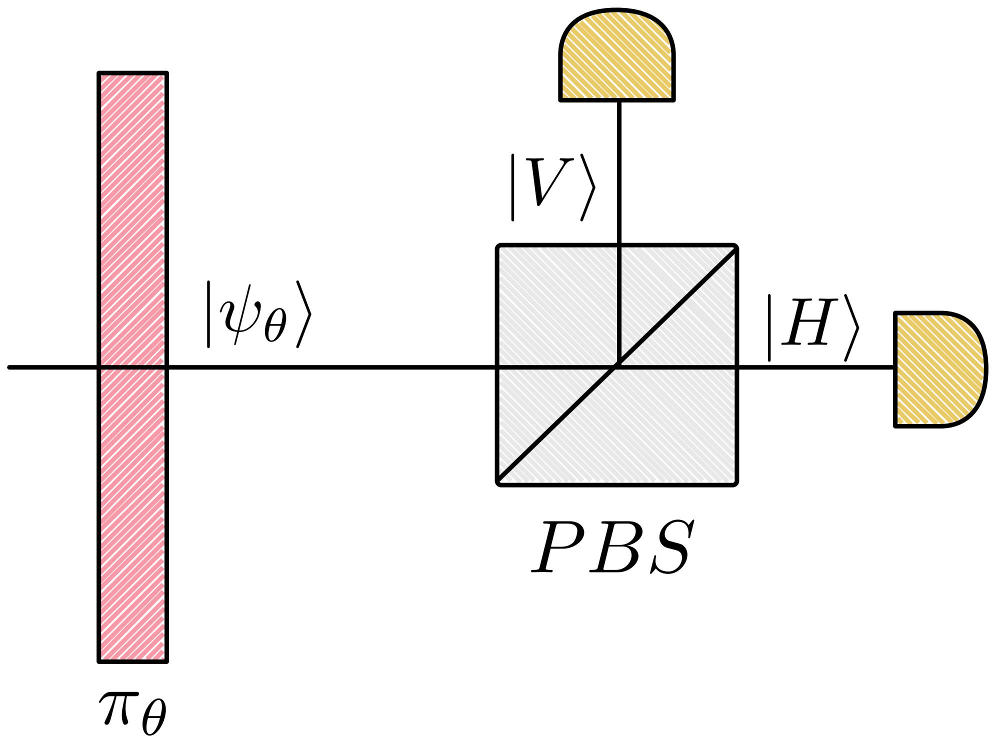
Thank you!

probability
quantum
ultimate
estimation
statistics
operator
Cauchy-Schwarz
POVM
variance
strategy
model
classical
efficient
QFI
experimenter
error
observable
parametrization
random
unbiased
Hamiltonian
estimator
Braunstein-Caves
sample-space
expectation

PARAMETER - DEPENDENT POVMS CAN ARISE IN MANY WAYS:

- estimating algebra deformation induced by a minimal length at Planck scale
- energy measurement in a Hamiltonian parameter estimation problem
- parameter - dependent interaction Hamiltonian between system and meter
- parameter characterizing the measurement apparatus

In all these cases, B - C inequality can be violated



OPTIMAL MEASUREMENT SCHEME ACCORDING TO STANDARD QET

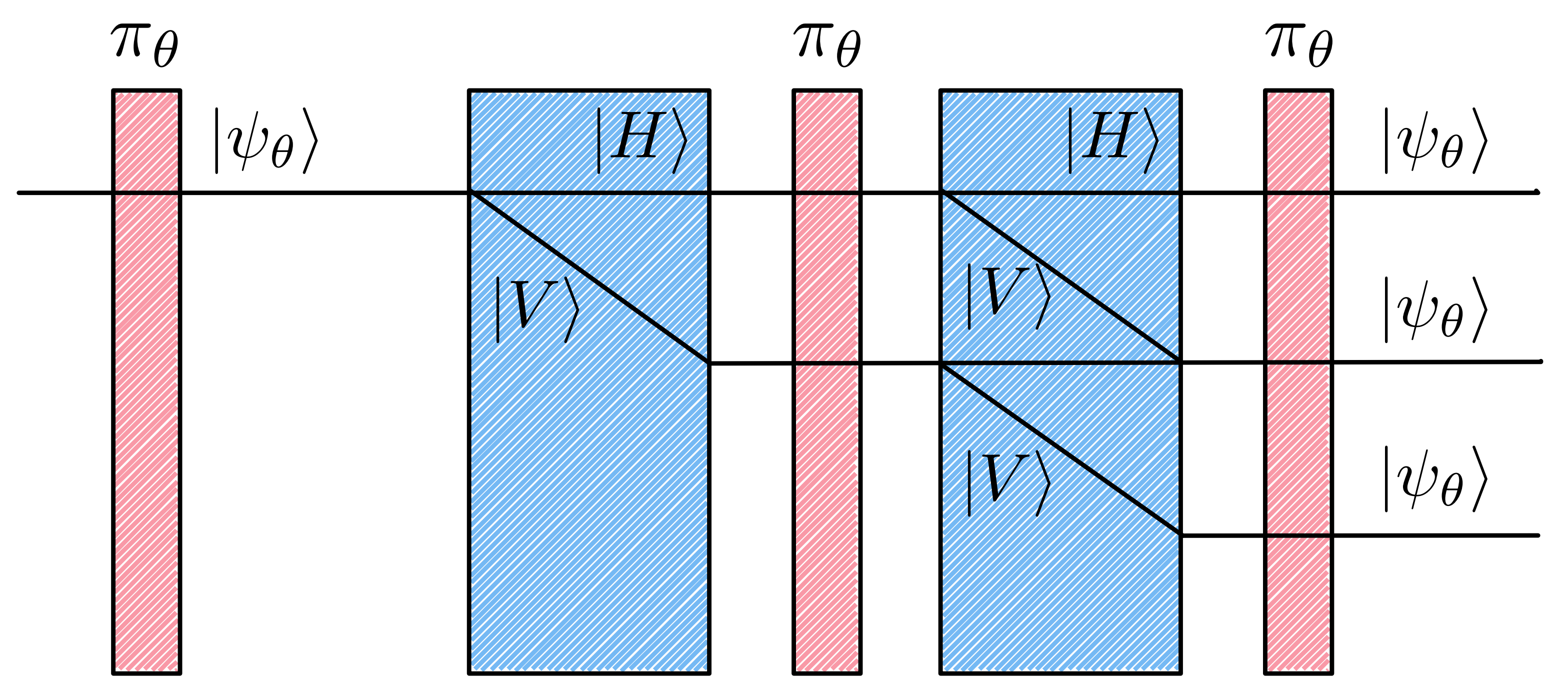
$$F_{PBS}(\theta) = J(\theta) = 4$$

$$|\psi\rangle_{t=0} = |f\rangle \otimes |\psi_\theta\rangle$$

$$|f\rangle = \int dx\, f(x)\, |x\rangle$$

$$|\psi_\theta\rangle = \cos\theta\, |H\rangle + \sin\theta\, |V\rangle$$

$$U = e^{-igP \otimes \pi_V}$$



QUANTUM CRAMER – RAO THEOREM CAN BE BEATEN

$\tilde{F}_{prot}(\theta)/J(\theta)$